On the nonradiative and quasistatic conditions and the limitations of circuit theory

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Field theory is used to analyze a simple circuit and to deduce the conditions for nonradiative and quasistatic fields. From the condition for quasistatic fields, which is the more stringent of the two, Kirchhoff’s voltage equation and its range of validity are deduced. These conditions for the validity of circuit theory are not treated appropriately in the undergraduate curricula. A simple numerical problem is given to illustrate these ideas. © 2007 American Association of Physics Teachers.

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I. INTRODUCTION

The undergraduate curricula in electromagnetism is based on Kirchhoff’s laws, which along with Ohm’s law, provide a set of rules that govern the values of the voltages and currents in all circuits. The limitations of circuit theory are not appropriately covered in basic texts on electric circuits, basic electromagnetic theory, and physics. The current design of electronic circuits involves increasingly high frequencies and hence we should have a clear understanding of the limitations of circuit theory. In this paper we will use the field theory from which circuit theory is derived to illustrate the limitations of the latter.

This paper is organized as follows. Section II introduces a simple electric circuit problem in the form of a mesh with four elements connected in series. In Sec. III the conditions for nonradiative and quasistatic fields due to the distribution of field sources are derived. This section also gives the circuit analysis from Maxwell’s equations, from which Kirchhoff’s voltage law is deduced, assuming that the magnetic field satisfies the quasistatic condition. In Sec. IV a numerical experiment is proposed that allows us to estimate the upper frequency limit of validity for Kirchhoff’s voltage law.

II. PROBLEM DESCRIPTION

The schematic diagram shown in Fig. 1(a) corresponds to a simple circuit. A concrete realization of this circuit is shown in Fig. 1(b). Kirchhoff’s law for the circuit in Fig. 1(a) gives

\[ \sum_i V_i = 0, \]

where \( V_i \) is the voltage across the \( i \)th element of the circuit (the voltage drop across the source is negative).

Any pair of conductors will have capacitance and any conductor with an electric current will have inductance. These reactive properties, which are distributed along any circuit, are called parasitic capacitance and parasitic inductance, respectively. Although these parasitic properties usually have small values and can be neglected at low frequency, they become gradually apparent as the frequency increases. As the voltage generator frequency increases, radiative effects also become more important. Both effects cause Kirchhoff’s voltage law to fail above a given value of the angular frequency. How can this frequency threshold be estimated?

III. DESCRIPTION OF THE SOLUTION

We begin with Maxwell’s equation for harmonic time-dependent fields:

\[ \nabla \times E = -i \omega B, \]

where \( \omega \) is the angular frequency. An integral form of Eq. (2) is

\[ \oint_{\Gamma} E \, d\ell = -i \omega \int_{S} B \, ds, \]

where \( \Gamma \) is the path of the conduction current and \( S(\Gamma) \) is any surface delimited by \( \Gamma \).

The magnetic field is given by

\[ B = \nabla \times A, \]

where the vector potential \( A \) is related to the current in the circuit by

\[ A = \frac{\mu}{4\pi} \oint_{\Gamma(S)} \frac{I(\ell') e^{-i\kappa R}}{R} \, d\ell'. \]

The source and observation point position vectors are \( r \) and \( r' \), respectively, \( R = r - r' \), and \( a_R = R/R \). Also \( I(\ell') \) is the electric current, with \( \ell' \) a longitudinal variable defined along the circuit \( \Gamma \) (see Fig. 2), and \( \kappa = 2\pi/\mu e \) is the wave number. We substitute Eq. (5) into Eq. (4) and obtain

\[ B = \nabla \times \left( \frac{\mu}{4\pi} \oint_{\Gamma(S)} \frac{I(\ell') d\ell' e^{-i\kappa R}}{R} \right). \]

Because derivatives in the operator \( \nabla \times \) apply to observation points \( r \) and the integral operator applies to source points \( r' \), the order of these operators can be interchanged. In Eq. (6) only observation points outside the source domain are of practical interest. In this way we find...
very distant from the circuit \( R \to \infty \), and the induction field \( B_I \), which is dominant at points very close to the circuit \( R \to 0 \).

The radiation field in the circuit’s far zone can be expressed as:

\[
B_R = \frac{i \kappa \mu}{4 \pi} \int_{(S)} I(\ell') \, d\ell' \times \frac{e^{-i\kappa R}}{R} a_R
\]

where \( \kappa \) is the angle between vectors \( r' \) and \( a_r \), so that \( r' \cdot a_r = r' \cos \kappa \).

If the geometry of the circuit is appropriately designed, the radiation field expressed by Eq. (8c) can be suppressed. By an appropriate design, we mean that the distance between each element of current and its image is small compared to \( \lambda \). Figure 3 illustrates this idea where a current element \( I(\ell_1') \) and its image (or return) \( I(\ell_2') \) are drawn. The differential contribution \( dB_R \) from both current elements to the total radiation field \( B_R \) at a given field point far away from the circuit is

\[
dB_R(r) \propto I(\ell_1') d\ell' e^{i\kappa r} \cos \alpha_1 + I(\ell_2') d\ell' e^{i\kappa r} \cos \alpha_2.
\]

We define \( \Delta r' = r_1' - r_2' \), where \( \Delta r' \) is the distance between the current element and its image (see Fig. 3), and find that in the limit \( \Delta r'/\lambda \to 0 \)
\[ dB_R \propto I(\ell') \, d\ell' \left( e^{i\sigma_1'} \cos a_1 - e^{i\sigma_2'} \cos a_2 \right) \]
\[ \propto I(\ell') \, d\ell' \left( e^{i\sigma_1'} \, \sigma_1 - e^{i\sigma_2'} \, a_2 \right) \]  
(10a)
\[ = 0, \]  
(10c)

where \( \lambda \) corresponds to the free space wavelength for a given frequency.

If the condition \( \Delta r' / \lambda \to 0 \) is fulfilled for each pair of current elements so that

\[ \max(\Delta r') \approx \lambda, \]  
(11)

it follows that the circuit radiation field \( B_R \) can be considered negligible. The inequality (11) is known as the nonradiative condition of the circuit.

The suppression of the radiation field does not imply that the largest dimension of the circuit must necessarily be very small compared to \( \lambda \), as in transmission lines. It is necessary to have any pair of current elements physically close together to avoid any radiation, but the circuit can be very large in comparison to \( \lambda \).

For a good design in the sense indicated, the magnetic field associated with the circuit is purely inductive:

\[ B = B_I = \frac{\mu}{4\pi \int_{\Gamma(S)} I(\ell') \, d\ell' \times e^{-i(2\pi/\lambda)R}}. \]  
(12)

If the largest circuit dimension, \( \sqrt{2D} \) in Fig. 1(b), is \( D_{\text{max}} \), and if \( D_{\text{max}} \) is much smaller than \( \lambda \), then \( 2\pi R/\lambda \to 0 \) at any point very close to the circuit. Thus, it follows that

\[ D_{\text{max}} \ll \lambda. \]  
(13)

The ratio \( D_{\text{max}} / \lambda \) is called the largest relative circuit dimension. If \( D_{\text{max}} / \lambda \ll 1 \), then \( R/\lambda \) at all points in the circuit is so small that \( e^{-i2\pi R/\lambda} \) is order one, and thus

\[ e^{-i(2\pi/\lambda)R} \approx \frac{a_R}{R^2}. \]  
(14)

The magnetic field phase is almost the same at all points of the circuit, \( 2\pi R'/\lambda \approx 0 \), and, consequently, the current is uniform throughout the circuit, that is, \( I(\ell') = I \) for all \( \ell' \). The condition (13) is the quasistatic condition of the field \( B \) and allows us to treat \( B \) as a magnetostatic field:

\[ B = \frac{\mu}{4\pi \int_{\Gamma(S)} I(\ell') \, d\ell' \times \frac{a_R}{R^2}}. \]  
(15)

The quasistatic condition is more restrictive than the nonradiation condition in Eq. (11), because (13) implies (11), but not the converse.

If we calculate the circulation of field \( E \) throughout the circuit of Fig. 1(b) assuming the quasistatic condition, we obtain

\[ \oint_{\Gamma(S)} E \, d\ell = -i\omega \int_{\Sigma} B \, ds \]  
(16a)
\[ \sum_i V_i = -i\omega \int_{\Sigma} \left( \frac{\mu}{4\pi} \int_{\Gamma(S)} \Gamma(\ell') \, d\ell' \times \frac{a_R}{R^2} \right) \, ds, \]  
(16b)
\[ \sum_i V_i + iZ_{L_p} = 0, \]  
(17)

where \( L_p \) is the inductance of the circuit and \( Z_{L_p} \) is the impedance associated with the frequency \( \omega \).

We reorder the terms of Eq. (16c) and again obtain Kirchhoff’s law:

\[ \sum_i V_i + iZ_{L_p} = 0, \]  
(17)

which models a circuit whose schematic diagram is shown in Fig. 4.

If we compare Figs. 1(a) and 4 we conclude that unless the term \( iZ_{L_p} \) can be neglected in comparison with \( \Sigma_i V_i \), the physical implementation of the circuit that is shown in Fig. 1(b) no longer corresponds to the schematic diagram given in Fig. 1(a).

IV. NUMERICAL EXPERIMENT

The right-hand side of Eq. (16c) models what is called a parasitic inductance. Because parasitic effects increase with frequency, the actual circuit of Fig. 1(b) is less well modeled by the schematic diagram of Fig. 1(a), and Eq. (1) becomes less precise. In this sense Kirchhoff’s voltage law fails. To illustrate this idea, a numerical experiment will be performed to estimate the angular frequency at which these effects become measurable. Equation (17) no longer matches the circuit of the schematic diagram of Fig. 1(a) unless the term \( iZ_{L_p} \) can be neglected in comparison to the sum \( \Sigma_i V_i \). How small the term \( iZ_{L_p} \) must be in order to be neglected depends on the application.

The idea is to estimate numerically the impedance \( Z_{L_p} \) as a function of frequency, compare the terms \( IR \) and \( |Z_{L_p}| \), and
estimate the frequency \( f^* \) at which Eq. (17) begins to be no longer approximated by Eq. (1) based on the following criteria:

\[
\sum_i V_i + IZ_{L_p} = \sum_i V_i, 
\]

for all \( f \leq f^* \), and

\[
2\pi f^* L_p = 0.2R. 
\]

That is, we will adopt an arbitrary numerical value of \( |Z_{L_p}| \), say 20% of \( R \), at which parasitic effects become important.

To implement the experiment, Eq. (16a) is discretized. The current circuit curve is divided into \( N \) rectilinear segments \( \Delta \ell_n \), and the surface coplanar to the circuit is divided into \( M \) surface elements \( \Delta s_m \), as shown in Fig. 5.

Based on Fig. 5 the following approximations are introduced: \( d\ell \rightarrow \Delta \ell_n a_\ell \), \( ds \rightarrow \Delta s_m \), \( f(S) \rightarrow \sum_{n=1}^{N} \), \( S \rightarrow \sum_{m=1}^{M} \).

Equation (16a) can then be approximated by

\[
\sum_i V_i = -li\omega \int_S \left( \frac{\mu}{4\pi} \oint_{f(S)} d\ell \times \frac{a_R}{R^2} \right) ds 
\]

\[
= -li\omega \frac{\mu}{4\pi} \sum_{m=1}^{M} \sum_{n=1}^{N} \Delta \ell_n \times \frac{a_{Rmn}}{R^2_{mn}} \Delta s_m 
\]

\[
= -li\omega \frac{\mu}{4\pi} \sum_{m=1}^{M} \sum_{n=1}^{N} \Delta \ell_n \Delta s_m 
\]

\[
\approx -li\omega \frac{\mu}{4\pi} \sum_{m=1}^{M} \sum_{n=1}^{N} \Delta s_m = L_p. 
\]

If the rectilinear and surface elements \( \Delta \ell_n \) and \( \Delta s_m \) are chosen to have equal size, \( \Delta \ell_n = \Delta \ell \) and \( \Delta s_m = \Delta s \), then Eq. (20c) can be simplified to

\[
\sum_i V_i = -li\omega \frac{\mu}{4\pi} \Delta \ell \Delta s \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{1}{R^2_{mn}}. 
\]

If \( \Delta s = \Delta s \times \Delta \ell \), we have

\[
L_p = \frac{\mu}{2\pi} \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{1}{R^2_{mn}}, 
\]

with \( h = \Delta \ell \).

To test Eq. (22) and our previous arguments, three realizations of the circuit in Fig. 5 were studied. These circuits are in the shapes of square and rectangular loops as shown in Fig. 6. Layout #1 in Fig. 6(a) and #3 in 6(c) are 4 cm \( \times \) 4 cm and 1 cm \( \times \) 1 cm square loops, respectively. Layout #2 is a 1 cm \( \times \) 4 cm rectangular loop [see Fig. 6(b)]. The internal inductances of these three layouts were estimated using Eq. (22) with \( h = 1 \) cm. The frequency \( f^* \), which is the
validity threshold of the criteria in Eqs. (18) and (19) with $R=50$ $\Omega$, was estimated. For this value of the frequency the maximum relative dimension of the circuit, $D_{\text{max}}/\lambda$, of each layout was also estimated. The results are shown in Table I.

From Table I it is observed that layout #1 exhibits the highest inductance ($L_p \sim 114.7$ nH) and the smallest threshold frequency $f^*$ ($f^* \sim 14$ MHz), and layout #3 exhibits the smallest inductance ($L_p \sim 16.7$ nH) and the highest threshold frequency $f^*$ ($f^* \sim 99.5$ MHz). Also, the largest relative circuit dimension for each circuit and for the criteria in Eqs. (18) and (19) is of the same order: $D_{\text{max}}/\lambda \sim 10^{-3}$.

From a frequency a little greater than 0 Hz to a value not estimated here above the threshold $f^*$, the magnetic field is quasistatic. However, for all three layouts, Kirchhoff’s voltage law, Eq. (1), is not exact at $f^*$ and above, according to Eqs. (18) and (19). Above this frequency Eq. (1) fails to predict the behavior of the circuit shown in Fig. 1(a).

From the results we see that the dimension of a circuit is important in determining in the applicability of circuit theory. Of the three physical implementations of the same circuit, layout #3, the smallest one, is the best because it satisfies the quasistatic condition for a wider range of frequencies. The deduction of Kirchhoff’s voltage law from Maxwell equations shows the intrinsic limitations of the former, which are related to the dimensions of the circuit. We have shown that for a given circuit a small distance between current elements suppresses radiation, and a small dimension makes the circuit behave as given by Kirchhoff’s voltage law.

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Table I. Estimated values of $L_p$, $f^*$, and $D_{\text{max}}/\lambda$. $L_p$ is the inductance of the circuit, $f^*$ is the frequency threshold of validity of Kirchhoff’s voltage law, and $D_{\text{max}}/\lambda$ is the largest circuit dimension, which is used as a reference to evaluate the quasistatic condition.

<table>
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<tr>
<th>Layout</th>
<th>$L_p$ (nH)</th>
<th>$f^*$ (MHz)</th>
<th>$D_{\text{max}}/\lambda$</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>114.7</td>
<td>14</td>
<td>0.0026</td>
</tr>
<tr>
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\[569\]